

Covariant formulations of BSSN and the standard gauge

J. David Brown

Department of Physics, North Carolina State University, Raleigh, NC 27695 USA

The BSSN and standard gauge equations are written in covariant form with respect to spatial coordinate transformations. The BSSN variables are defined as tensors with no density weights. This allows us to evolve a given set of initial data using two different coordinate systems and to relate the results using the familiar tensor transformation rules. Two variants of the covariant equations are considered. These differ from one another in the way that the determinant of the conformal metric is evolved.

I. INTRODUCTION

The BSSN formulation of Einstein's equations [1, 2] is in widespread use in the numerical relativity community. These equations are most often used in conjunction with the “standard gauge” conditions, namely, 1+log slicing and the Gamma-driver shift.

The BSSN variables include the conformal metric g_{ab} , conformal factor φ , and the trace and trace-free parts of the extrinsic curvature, K and A_{ab} . They are defined in terms of the physical spatial metric \hat{g}_{ab} and the physical extrinsic curvature \hat{K}_{ab} by

$$\hat{g}_{ab} = e^{4\varphi} g_{ab}, \quad (1a)$$

$$\hat{K}_{ab} = e^{4\varphi} (A_{ab} + g_{ab} K/3). \quad (1b)$$

The conformal metric is chosen to have unit determinant, $g = 1$, and A_{ab} is trace-free. A key ingredient of the BSSN formulation is the use of the “conformal connection functions”, defined by $\Gamma^a \equiv -\partial_b g^{ab}$.

The 1+log slicing condition is an evolution equation for the lapse function α that takes the form [3]

$$\partial_t \alpha = \beta^c \partial_c \alpha - 2\alpha K, \quad (2)$$

where β^a is the shift vector. The Gamma driver shift is defined by [4]

$$\partial_t \beta^a = \beta^c \partial_c \beta^a + \frac{3}{4} B^a, \quad (3a)$$

$$\partial_t B^a = \beta^c \partial_c B^a + \partial_t \Gamma^a - \beta^c \partial_c \Gamma^a - \eta B^a, \quad (3b)$$

where B^a is an auxiliary variable and η is a constant. The term $\partial_t \Gamma^a$ in Eq. (3b) is replaced with the right-hand side of the equation of motion for Γ^a .

The Gamma driver shift condition is not generally covariant. In other words, Eqs. (3) do not preserve their form under a time-independent transformation of the spatial coordinates. To see this, note that β^a and $\partial_t \beta^a$ transform as contravariant vectors. The advection term $\beta^c \partial_c \beta^a$ in Eq. (3a) does not transform as a contravariant vector. This spoils the covariance of Eq. (3a). Note, however, that the advection terms are not always included in the Gamma driver shift condition [5]. So for the moment let us ignore the terms with $\beta^c \partial_c$ acting on β^a , B^a , and Γ^a . Then Eq. (3a) shows that B^a and $\partial_t B^a$ should transform as contravariant vectors. However, the right-hand side of Eq. (3b) depends on the connection functions

Γ^a , which do not form a contravariant vector. Instead, Γ^a obeys a rather complicated transformation rule determined from the following considerations. The conformal metric has unit determinant, $g = 1$. This equation is generally covariant under spatial coordinate transformations only if g_{ab} is defined as a type $\binom{0}{2}$ tensor density of weight $-2/3$. This makes g a scalar, which is set equal to the scalar 1. Therefore the conformal connection functions $\Gamma^a \equiv -\partial_b g^{ab}$ transform as the contraction of the derivative of a type $\binom{2}{0}$ tensor density of weight $2/3$. This is a complicated transformation rule which I will not bother to write out in detail. The time derivative, $\partial_t \Gamma^a$, also satisfies this rule. We see that even if we ignore the advection terms, there is a mismatch in Eqs. (3) in the way that the individual terms transform under time-independent changes of spatial coordinates.

In addition to the conformal metric g_{ab} and the conformal connection functions Γ^a , the BSSN variables include φ , A_{ab} , and K . Because g_{ab} carries a nonzero density weight, the conformal factor φ must transform as the logarithm of a weight $1/6$ scalar density. The variable A_{ab} is a trace-free type $\binom{0}{2}$ tensor density of weight $-2/3$. The trace of the extrinsic curvature K is a scalar. With these transformation rules, the BSSN equations and the 1+log slicing conditions are covariant.

In section II, I discuss the issue of general covariance in more detail. This serves as further motivation for the subsequent analysis. In Section III, I rewrite the BSSN equations in terms of simple tensor variables with no density weights. One of the key steps in the analysis is the introduction of a background connection, as suggested by Garfinkle, Gundlach, and Hilditch [6]. Another key step is to recognize that the condition $g = 1$ should be replaced by an evolution equation for g [7].

There are two natural choices for the evolution of g , which are presented in Section IV. One is the “Lagrangian case” in which $\partial_t g = 0$. Then g is constant in time, equal to its initial value. If the initial value of g is unity and the background connection vanishes, these equations reduce to the traditional BSSN equations. These equations are not strongly hyperbolic unless the trace-free property of the variable A_{ab} is actively enforced during the evolution. As an alternative, one can add a term proportional to A to the evolution equation for the conformal metric. This yields a strongly hyperbolic system without the need to actively enforce the con-

dition $A = 0$.

Another choice for the evolution of g is $\partial_t g = \mathcal{L}_\beta g$, where \mathcal{L} is the Lie derivative. This is the “Eulerian case”. The BSSN equations in Eulerian form contain fewer terms than the traditional BSSN equations.

In Section III the tensorial BSSN variables are used to rewrite the Gamma-driver shift condition in generally covariant form. If the initial value of g is unity and the background connection vanishes, the covariant Gamma-driver shift equations for the Lagrangian case reduce to the familiar Eqs. (3). As an alternative, the shift condition can be defined using Eqs. (3) with the Eulerian evolution equation for $\partial_t \Gamma^a$.

II. SPATIAL COVARIANCE IN NUMERICAL RELATIVITY

Let me discuss the issue of spatial covariance in concrete terms. Let’s say we are given a physical metric and extrinsic curvature, \hat{g}_{ab} and \hat{K}_{ab} , that satisfy the Hamiltonian and momentum constraints. For simplicity, let us assume that these tensors \hat{g}_{ab} and \hat{K}_{ab} are expressed in terms of a single coordinate patch with “Cartesian” coordinates x, y , and z . By calling the coordinates Cartesian I mean that each coordinate ranges over an interval of the real number line, with no periodic identification and no coordinate singularities. In this case we can transform to “spherical” coordinates r, θ , and ϕ using the familiar relations $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, and $z = r \cos \theta$. Let us denote the initial data in spherical coordinates by \hat{g}'_{ab} , \hat{K}'_{ab} . The initial data in these two coordinate systems are related by

$$\hat{g}'_{ab} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} \hat{g}_{cd}, \quad (4a)$$

$$\hat{K}'_{ab} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} \hat{K}_{cd}, \quad (4b)$$

where x^a are Cartesian coordinates and x'^a are spherical coordinates.

We want to evolve these data using the BSSN system. Starting with the physical metric and extrinsic curvature in Cartesian coordinates, \hat{g}_{ab} and \hat{K}_{ab} , we apply the definitions (1) to obtain initial values for the BSSN variables g_{ab} , φ , A_{ab} , K , and Γ^a . Alternatively, we can start with the physical metric and extrinsic curvature in spherical coordinates, \hat{g}'_{ab} and \hat{K}'_{ab} , and define the BSSN variables g'_{ab} , φ' , A'_{ab} , K' , and Γ'^a . The primed and unprimed BSSN variables will be related by the coordinate transformation rules outlined in the introduction. For example, the relation for the conformal metric is

$$g'_{ab} = \left| \frac{\partial x}{\partial x'} \right|^{-2/3} \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{cd}. \quad (5)$$

The factor $|\partial x / \partial x'|$ is the Jacobian of the coordinate transformation.

In order to evolve these data we must choose a lapse function α and shift vector β^a . Let us assume for definiteness that the lapse and shift are determined by evolution type equations and, for the moment, let us assume that these equations are spatially covariant. For example, one might consider 1+log slicing and a “modified” Gamma driver condition obtained by replacing the term $\partial_t \Gamma^a - \beta^c \partial_c \Gamma^a$ with, say, $\hat{D}_b \Sigma^{ab}$, and replacing the remaining spatial derivatives with covariant derivatives. (Here, \hat{D}_b is the physical spatial covariant derivative and $\Sigma^{ab} \equiv (\hat{D}^a \beta^b - \alpha \hat{K}^{ab})^{TF}$ is the distortion tensor. TF denotes the trace-free part.)

These gauge conditions require us to specify initial values for the lapse, shift, and auxiliary variable. Let α , β^a , and B^a denote these initial values in Cartesian coordinates. The initial values in spherical coordinates are related by the familiar tensor transformation rules: For the scalar lapse we have $\alpha' = \alpha$, and for the contravariant vector shift,

$$\beta'^a = \frac{\partial x'^a}{\partial x^b} \beta^b. \quad (6)$$

The auxiliary variable B^a also transforms as a contravariant vector.

Now we’re ready to evolve both the “unprimed” Cartesian coordinate data and the “primed” spherical coordinate data from the initial time t_i to some final time t_f . How do the primed and unprimed BSSN variables compare at t_f ? If, as we assumed above, the lapse and shift are determined by covariant relations, then the two sets of BSSN variables at time t_f will be related by the same transformation rules that apply to the initial data. In particular, the two conformal metrics at time t_f will be related by Eq. (5).

This, of course, is a good situation. We would like to have the option of evolving our initial data using different spatial coordinate systems, and we would like to be able to compare the results. But the Gamma driver shift condition Eq. (3) is not covariant. Thus, at times $t > t_i$, the shift vector obtained from the Cartesian coordinate evolution will differ *geometrically* from the shift vector obtained from the spherical coordinate evolution. These two shift vectors will not be related by a coordinate transformation; rather, they will be geometrically distinct vector fields.¹

Because the shift vector depends on the coordinate system, the BSSN variables at time t_f will *not* be related by the coordinate transformation rules outlined above. In particular, the conformal metrics at time t_f will not be related by Eq. (5). We can recombine the BSSN variables to form the physical metric and physical extrinsic

¹ With the 1+log condition (2), the slicing of spacetime does not depend on the shift vector or the coordinate system. If the advection term $\beta^c \partial_c \alpha$ is dropped, then a non-covariant shift condition will cause the slicing to depend on the coordinate system.

curvature. The physical tensors will not be related by the transformation (4).

This shortcoming of the BSSN formulation with Gamma-driver shift can be fixed. This is done by rewriting the BSSN equations and the standard gauge conditions in terms of regular tensors with no density weights. The density weights are removed by eliminating the requirement that the determinant of the conformal metric, g , should equal 1 [7]. It is then necessary to specify an evolution equation for g . The simplest choice is to let g be constant in time. Then g is determined by its initial value, a weight 2 scalar density called \bar{g} . The conformal metric g_{ab} is then a type $\binom{0}{2}$ tensor with no density weight. The next step is to define a variable $\Lambda^a = g^{bc}(\Gamma_{bc}^a - \tilde{\Gamma}_{bc}^a)$ to take the place of the conformal connection functions [6]. Here, Γ_{bc}^a are the Christoffel symbols constructed from the conformal metric, and $\tilde{\Gamma}_{bc}^a$ is a background connection.

The BSSN and standard gauge equations can be written in terms of the regular tensors g_{ab} , φ , A_{ab} , K , and Λ^a . These equations reduce to the traditional forms in use by numerical relativity groups when the density \bar{g} is set to unity and the background connection is that of a flat metric in Cartesian coordinates: $\tilde{\Gamma}_{bc}^a = 0$. If we want to transform an initial data set from Cartesian to spherical coordinates, and preserve the tensor transformation rules under evolution, then we must transform \bar{g} as a weight 2 density and $\tilde{\Gamma}_{bc}^a$ as a connection. In particular, in spherical coordinates, \bar{g} would no longer be 1 and $\tilde{\Gamma}_{bc}^a$ would no longer vanish.

III. TENSOR VARIABLES FOR BSSN

In this section I derive the BSSN equations from scratch using only tensors with no density weights. Begin with the “gdot-Kdot” form of the Einstein evolution equations:

$$\partial_\perp \hat{g}_{ab} = -2\alpha \hat{K}_{ab} , \quad (7a)$$

$$\begin{aligned} \partial_\perp \hat{K}_{ab} &= \alpha \hat{K} \hat{K}_{ab} - 2\alpha \hat{K}_{ac} \hat{K}_b^c \\ &\quad + \alpha \hat{R}_{ab} - \hat{D}_a \hat{D}_b \alpha . \end{aligned} \quad (7b)$$

The time derivative operator is defined by $\partial_\perp \equiv \partial_t - \mathcal{L}_\beta$ where \mathcal{L}_β is the Lie derivative along the shift. The Hamiltonian and momentum constraints are

$$\mathcal{H} \equiv \hat{K}^2 - \hat{K}_{ab} \hat{K}^{ab} + \hat{R} , \quad (8a)$$

$$\mathcal{M}_a \equiv \hat{D}_b (\hat{K}_a^b - \hat{K} \delta_a^b) . \quad (8b)$$

Indices on the momentum constraint and extrinsic curvature are raised and lowered with the physical metric.

The BSSN variables are defined by Eqs (1). However, we will not assume any restrictions on the determinant of g_{ab} or the trace of A_{ab} . Then these definitions can be inverted to obtain

$$g_{ab} = (\hat{g}/g)^{-1/3} \hat{g}_{ab} , \quad (9a)$$

$$A_{ab} = (\hat{g}/g)^{-1/3} (\hat{K}_{ab} - \hat{g}_{ab} \hat{K}/3 + \hat{g}_{ab} A/3) , \quad (9b)$$

$$\varphi = \frac{1}{12} \ln(\hat{g}/g) , \quad (9c)$$

$$K = \hat{K} - A . \quad (9d)$$

Note that $A \equiv g^{ab} A_{ab}$ and $\hat{K} \equiv \hat{g}^{ab} \hat{K}_{ab}$. Defined in this way, the BSSN variables g_{ab} , A_{ab} , φ , and K are tensors with no density weights.

Now compute the time derivatives of Eqs. (9) using the gdot-Kdot equations (7), then use Eqs. (1) to express the results in terms of BSSN variables. This is a straightforward, although somewhat tedious calculation. It is useful to note that $(\hat{g}/g)^{1/3} = e^{4\varphi}$, and for the last two terms in Eq. (7b),

$$\begin{aligned} \hat{R}_{ab} &= R_{ab} - 2D_a D_b \varphi + 4D_a \varphi D_b \varphi \\ &\quad - 2g_{ab} (D^2 \varphi + 2D^c \varphi D_c \varphi) , \end{aligned} \quad (10a)$$

$$\begin{aligned} \hat{D}_a \hat{D}_b \alpha &= D_a D_b \alpha - 4D_{(a} \alpha D_{b)} \varphi \\ &\quad + 2g_{ab} D^c \alpha D_c \varphi . \end{aligned} \quad (10b)$$

The result of this calculation is

$$\partial_\perp g_{ab} = \frac{1}{3} g_{ab} \partial_\perp \ln g - 2\alpha A_{ab} + \frac{2}{3} \alpha g_{ab} A , \quad (11a)$$

$$\begin{aligned} \partial_\perp A_{ab} &= \frac{1}{3} A_{ab} \partial_\perp \ln g + \frac{1}{3} g_{ab} \partial_\perp A - 2\alpha A_{ac} A_b^c + \alpha A_{ab} K + \frac{1}{3} \alpha A (5A_{ab} - Ag_{ab} - Kg_{ab}) \\ &\quad + e^{-4\varphi} [-2\alpha D_a D_b \varphi + 4\alpha D_a \varphi D_b \varphi + 4D_{(a} \alpha D_{b)} \varphi - D_a D_b \alpha + \alpha R_{ab}]^{\text{TF}} , \end{aligned} \quad (11b)$$

$$\partial_\perp \varphi = -\frac{1}{12} \partial_\perp \ln g - \frac{1}{6} \alpha (K + A) , \quad (11c)$$

$$\partial_\perp K = -\partial_\perp A + \alpha (K + A)^2 + e^{-4\varphi} (\alpha R - 8\alpha D^a \varphi D_a \varphi - 8\alpha D^2 \varphi - D^2 \alpha - 2D^a \alpha D_a \varphi) . \quad (11d)$$

The superscript “TF” denotes the trace-free part of the

expression in brackets.

Now define

$$\Delta\Gamma_{bc}^a \equiv \Gamma_{bc}^a - \tilde{\Gamma}_{bc}^a, \quad (12a)$$

$$\Delta\Gamma^a \equiv g^{bc}\Delta\Gamma_{bc}^a, \quad (12b)$$

where $\tilde{\Gamma}_{bc}^a$ is a background connection. Although it is not necessary, it is convenient to think of the background connection as being built from a background metric, \tilde{g}_{ab} . I assume that the background connection is time independent. Note that $\Delta\Gamma^a$ is a contravariant vector.

From the definition of the Riemann tensor, we have the following identity:

$$\begin{aligned} R_{ab} \equiv & -\frac{1}{2}g^{cd}\tilde{D}_c\tilde{D}_d g_{ab} + g_{c(a}\tilde{D}_{b)}\Delta\Gamma^c \\ & -g^{cd}g_{e(a}\tilde{R}_{b)cd}{}^e + g^{de}\Delta\Gamma_{de}^c\Delta\Gamma_{(ab)c} \\ & +g^{cd}\left(2\Delta\Gamma_{c(a}^e\Delta\Gamma_{b)ed} + \Delta\Gamma_{ac}^e\Delta\Gamma_{ebd}\right). \end{aligned} \quad (13)$$

Here, \tilde{D}_a is the covariant derivative and $\tilde{R}^a{}_{bcd}$ is the Riemann tensor built from $\tilde{\Gamma}_{bc}^a$.

The equations of motion (11) imply

$$\begin{aligned} \partial_\perp(\Delta\Gamma^a) = & g^{bc}\tilde{D}_b\tilde{D}_c\beta^a - g^{bc}\tilde{R}^a{}_{bcd}\beta^d - 2A^{ab}\partial_b\alpha \\ & -\frac{2\alpha}{\sqrt{g}}\tilde{D}_b(\sqrt{g}A^{ab}) - \frac{1}{3}\Delta\Gamma^a(\partial_\perp \ln g + 2\alpha A) \\ & -\frac{1}{6}g^{ab}\partial_b(\partial_\perp \ln g - 4\alpha A). \end{aligned} \quad (14)$$

Again, this is a straightforward, but tedious calculation. We now let Λ^a denote a new BSSN variable which equals $\Delta\Gamma^a$ when the following constraint holds:

$$\mathcal{C}^a \equiv \Lambda^a - \Delta\Gamma^a. \quad (15)$$

Λ^a is a contravariant vector.

Next, we modify the equations of motion using the Hamiltonian and momentum constraints. The Hamiltonian constraint is $\mathcal{H} = 0$; from Eq. (8a) we find

$$\begin{aligned} \mathcal{H} = & \frac{2}{3}(K + A)^2 + \frac{1}{3}A^2 - A_{ab}A^{ab} \\ & + e^{-4\varphi}(R - 8D^a\varphi D_a\varphi - 8D^2\varphi). \end{aligned} \quad (16)$$

Now add $-\alpha\mathcal{H}$ to the right-hand side of Eq. (11d). The momentum constraint is $\mathcal{M}^a = 0$; from Eq. (8b) we find

$$\begin{aligned} \mathcal{M}^a = & \frac{1}{\sqrt{g}}e^{-4\varphi}\tilde{D}_b(\sqrt{g}A^{ab}) + 6e^{-4\varphi}(A^{ab} - Ag^{ab}/3)\partial_b\varphi \\ & - e^{-4\varphi}g^{ab}\partial_b(2K/3 + A) + e^{-4\varphi}A^{bc}\Delta\Gamma_{bc}^a. \end{aligned} \quad (17)$$

The equation of motion for Λ^a is obtained by replacing $\partial_\perp\Delta\Gamma^a$ on the left-hand side of Eq. (14) with $\partial_\perp\Lambda^a$. We then add $2\alpha e^{4\varphi}\mathcal{M}^a$ to the right-hand side of this equation.

The analysis described above yields the following equations for the tensor BSSN variables:

$$\partial_\perp g_{ab} = \frac{1}{3}g_{ab}\partial_\perp \ln g - 2\alpha A_{ab} + \frac{2}{3}\alpha g_{ab}A, \quad (18a)$$

$$\begin{aligned} \partial_\perp A_{ab} = & \frac{1}{3}A_{ab}\partial_\perp \ln g + \frac{1}{3}g_{ab}\partial_\perp A - 2\alpha A_{ac}A_b^c + \alpha A_{ab}K + \frac{1}{3}\alpha A(5A_{ab} - Ag_{ab} - Kg_{ab}) \\ & + e^{-4\varphi}[-2\alpha D_a D_b\varphi + 4\alpha D_a\varphi D_b\varphi + 4D_{(a}\alpha D_{b)}\varphi - D_a D_b\alpha + \alpha\mathcal{R}_{ab}]^{\text{TF}}, \end{aligned} \quad (18b)$$

$$\partial_\perp\varphi = -\frac{1}{12}\partial_\perp \ln g - \frac{1}{6}\alpha(K + A), \quad (18c)$$

$$\partial_\perp K = -\partial_\perp A + \frac{\alpha}{3}(K^2 + 2KA) + \alpha A_{ab}A^{ab} - e^{-4\varphi}(D^2\alpha + 2D^a\alpha D_a\varphi), \quad (18d)$$

$$\begin{aligned} \partial_\perp\Lambda^a = & g^{bc}\tilde{D}_b\tilde{D}_c\beta^a - g^{bc}\tilde{R}^a{}_{bcd}\beta^d - \frac{1}{3}\Delta\Gamma^a\partial_\perp \ln g - \frac{1}{6}g^{ab}\partial_b\partial_\perp \ln g \\ & - 2(A^{bc} - g^{bc}A/3)(\delta_b^a\partial_c\alpha - 6\alpha\delta_b^a\partial_c\varphi - \alpha\Delta\Gamma_{bc}^a) - \frac{4}{3}\alpha g^{ab}\partial_b K, \end{aligned} \quad (18e)$$

where

$$\mathcal{R}_{ab} \equiv -\frac{1}{2}g^{cd}\tilde{D}_c\tilde{D}_d g_{ab} + g_{c(a}\tilde{D}_{b)}\Lambda^c - g^{cd}g_{e(a}\tilde{R}_{b)cd}{}^e + g^{de}\Delta\Gamma_{de}^c\Delta\Gamma_{(ab)c} + g^{cd}\left(2\Delta\Gamma_{c(a}^e\Delta\Gamma_{b)ed} + \Delta\Gamma_{ac}^e\Delta\Gamma_{ebd}\right). \quad (19)$$

In Eq. (19), \mathcal{R}_{ab} is defined by using Λ^a in place of $\Delta\Gamma^a$ in the identity (13).

The Eqs. (18) are not complete evolution equations because $\partial_\perp g$ and $\partial_\perp A$ appear on the right-hand sides. These equations are consistent in the sense that we can

use them to compute $\partial_\perp g$, and the result is an identity: $\partial_\perp g = \partial_\perp g$. Similarly, Eqs. (18) yield an identity for $\partial_\perp A$.

Because the quantities $\partial_\perp g$ and $\partial_\perp A$ appear on the right-hand sides of Eqs. (18), we must specify how g and

A evolve. There are two natural choices for g , namely, $\partial_t g = 0$ and $\partial_\perp g = 0$. In Ref. [7, 8], these were referred to as the *Lagrangian case* and the *Eulerian case*, respectively. For A , I will only consider the evolution equation $\partial_t A = 0$. These cases are described in detail in the next section.

Using the tensorial variables define above, the standard gauge conditions are:

$$\partial_t \alpha = \beta^c \tilde{D}_c \alpha - 2\alpha K, \quad (20a)$$

$$\partial_t \beta^a = \beta^c \tilde{D}_c \beta^a + \frac{3}{4} B^a, \quad (20b)$$

$$\partial_t B^a = \beta^c \tilde{D}_c B^a + (\partial_t \Lambda^a - \beta^c \tilde{D}_c \Lambda^a) - \eta B^a. \quad (20c)$$

Equation (20a) is the 1+log slicing condition and Eqs. (20b,c) are the Gamma-driver shift condition. The extra variable B^a is a contravariant vector with no density weight. On the right-hand side of Eq. (20c) the term $\partial_t \Lambda^a$ is eliminated using the BSSN equation of motion for Λ^a .

IV. LAGRANGIAN AND EULERIAN CASES

For the Lagrangian case we have $\partial_t g = 0$ and $\partial_t A = 0$. For any choice of initial values, g and A will remain unchanged throughout the evolution. Let us call these ini-

tial values \bar{g} and \bar{A} . Thus \bar{g} is a time independent spatial scalar density of weight +2, and \bar{A} is a time independent spatial scalar. The time independence of g and A imply $\partial_\perp \ln g = -2D_a \beta^a$ and $\partial_\perp A = -\beta^a \partial_a A$.

Since g and A equal their initial values for all time, we can replace g with \bar{g} and A with \bar{A} wherever they appear in Eqs. (18). Note that the covariant divergence of the shift depends on the spatial metric only through its determinant: $D_a \beta^a = \partial_a(\sqrt{g}\beta^a)/\sqrt{g}$. Since the determinant is constant in time, we can replace $D_a \beta^a$ with $\bar{D}_a \beta^a$, where \bar{D}_a is the covariant derivative built from the initial conformal metric \bar{g}_{ab} . If we make the replacements $g \rightarrow \bar{g}$, $A \rightarrow \bar{A}$, and $D_a \beta^a \rightarrow \bar{D}_a \beta^a$ everywhere, we obtain the traditional BSSN equations written in covariant form.

The traditional BSSN equations are not strongly hyperbolic unless the algebraic constraint $A = \bar{A}$ is continuously enforced [9, 10]. (This is true for any choice of gauge conditions, not just the standard gauge.) In practice, the constraint $A = 0$ is imposed by making the replacement $A_{ab} \rightarrow A_{ab} - g_{ab}A/3$ after every sub-timestep in the numerical evolution. This prevents A from developing a non-zero value due to numerical error.

As an alternative, we can achieve strong hyperbolicity by leaving the term $2\alpha g_{ab}A/3$ in Eq. (18a) alone [10]. If we do this and also choose $\bar{A} = 0$, we find the following *Lagrangian BSSN equations*:

$$\partial_\perp g_{ab} = -\frac{2}{3} g_{ab} \bar{D}_c \beta^c - 2\alpha A_{ab} + \frac{2}{3} \alpha g_{ab} A, \quad (21a)$$

$$\begin{aligned} \partial_\perp A_{ab} = & -\frac{2}{3} A_{ab} \bar{D}_c \beta^c - 2\alpha A_{ac} A_b^c + \alpha A_{ab} K \\ & + e^{-4\varphi} [-2\alpha D_a D_b \varphi + 4\alpha D_a \varphi D_b \varphi + 4D_{(a} \alpha D_{b)} \varphi - D_a D_b \alpha + \alpha \mathcal{R}_{ab}]^{\text{TF}}, \end{aligned} \quad (21b)$$

$$\partial_\perp \varphi = \frac{1}{6} \bar{D}_c \beta^c - \frac{1}{6} \alpha K, \quad (21c)$$

$$\partial_\perp K = \frac{\alpha}{3} K^2 + \alpha A_{ab} A^{ab} - e^{-4\varphi} (D^2 \alpha + 2D^a \alpha D_a \varphi), \quad (21d)$$

$$\begin{aligned} \partial_\perp \Lambda^a = & \mathcal{C}^b \tilde{D}_b \beta^a + g^{bc} \tilde{D}_b \tilde{D}_c \beta^a - g^{bc} \tilde{R}^a_{bcd} \beta^d + \frac{2}{3} \Delta \Gamma^a \bar{D}_c \beta^c + \frac{1}{3} D^a \bar{D}_c \beta^c \\ & - 2A^{bc} (\delta_b^a \partial_c \alpha - 6\alpha \delta_b^a \partial_c \varphi - \alpha \Delta \Gamma_{bc}^a) - \frac{4}{3} \alpha g^{ab} \partial_b K. \end{aligned} \quad (21e)$$

\mathcal{R}_{ab} is given by Eq. (19). These equations with the standard gauge Eqs.(20) are strongly hyperbolic without the need to enforce $A = 0$ explicitly. If the term $2\alpha g_{ab}A/3$ is omitted from Eq. (21a), the result is the *traditional BSSN equations*. The traditional BSSN equations coincide with the equations that are in widespread use in the numerical relativity community when the background connection vanishes, $\tilde{\Gamma}_{bc}^a = 0$, and the initial data satisfies $\bar{g} = 1$.

It is a bit of an overstatement to say that the Lagrangian system (21), or the traditional system with

$A = 0$ enforced, is strongly hyperbolic. As shown by Beyer and Sarbach [11], the traditional BSSN system plus standard gauge is strongly hyperbolic for $2\alpha \neq e^{4\varphi}$. The condition $2\alpha \neq e^{4\varphi}$ is likely violated on a surface of co-dimension one in black hole simulations, but in practice this does not seem to be a problem.

Another natural choice for the evolution of g is the *Eulerian case*, $\partial_\perp g = 0$. Let us assume as before that A is time independent, $\partial_t A = 0$. Let us replace A with its initial value $\bar{A} = 0$ everywhere except in the $\partial_\perp g_{ab}$

equation. Then the *Eulerian BSSN equations* are

$$\partial_{\perp} g_{ab} = -2\alpha A_{ab} + \frac{2}{3}\alpha g_{ab} A, \quad (22a)$$

$$\partial_{\perp} A_{ab} = -2\alpha A_{ac} A_b^c + \alpha A_{ab} K + e^{-4\varphi} [-2\alpha D_a D_b \varphi + 4\alpha D_a \varphi D_b \varphi + 4D_{(a} \alpha D_{b)} \varphi - D_a D_b \alpha + \alpha \mathcal{R}_{ab}]^{\text{TF}}, \quad (22b)$$

$$\partial_{\perp} \varphi = -\frac{1}{6}\alpha K, \quad (22c)$$

$$\partial_{\perp} K = \frac{\alpha}{3} K^2 + \alpha A_{ab} A^{ab} - e^{-4\varphi} (D^2 \alpha + 2D^a \alpha D_a \varphi), \quad (22d)$$

$$\partial_{\perp} \Lambda^a = \mathcal{C}^b \tilde{D}_b \beta^a + g^{bc} \tilde{D}_b \tilde{D}_c \beta^a - g^{bc} \tilde{R}^a_{bcd} \beta^d - 2A^{bc} (\delta_b^a \partial_c \alpha - 6\alpha \delta_b^a \partial_c \varphi - \alpha \Delta \Gamma_{bc}^a) - \frac{4}{3} \alpha g^{ab} \partial_b K. \quad (22e)$$

Again, \mathcal{R}_{ab} is given by Eq. (19). These equations with the standard gauge Eqs. (20) are strongly hyperbolic for $2\alpha \neq e^{4\varphi}$ and do not require enforcement of the algebraic constraint $A = \bar{A} = 0$. They are more simple than both the Lagrangian equations (21) and the traditional BSSN equations.

Note that the Gamma driver shift (20b,c) depends on $\partial_t \Lambda^a$. This term is to be replaced with appropriate terms from the equation of motion for Λ^a . There are two possibilities. The term $\partial_t \Lambda^a$ can be defined using either the Lagrangian equation (21e) or the Eulerian equation (22e). If one wants the Gamma driver shift condition as it is currently defined in the numerical relativity community, then the Lagrangian equation should be used. This is the case even if one chooses to evolve the BSSN variables using the Eulerian Eqs. (22). The BSSN equations, either Lagrangian or Eulerian, with the standard gauge

that uses the Lagrangian equation for $\partial_t \Lambda^a$ to define the Gamma driver shift, is strongly hyperbolic for $2\alpha \neq e^{4\varphi}$.

It would be interesting to investigate the properties of the shift condition defined by using the Eulerian equation (22e) to eliminate $\partial_t \Lambda^a$ from Eqs. (20). In this case the BSSN equations (either Lagrangian or Eulerian) plus gauge conditions are strongly hyperbolic for $8\alpha \neq 3e^{4\varphi}$. A detailed analysis of hyperbolicity for these systems will be given elsewhere [10].

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